On cubic surface bundles

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April 19, 2024
Birational Geometry Seminar
Online seminar

Summary/plan

1. General question of interest: determine which smooth projective varieties X are **rational**: is X birational to $\mathbb{P}^n_k$? (or stably rational, or retract rational...)

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   - invariants: use Galois cohomology and geometry of cubics;
   - example:
     $X : xz^2u^3 + y^2zv^3 + xy^2w^3 + ft^3 = 0 \subset \mathbb{P}^2_{[x:y:z]} \times \mathbb{P}^3_{[u:v:w:t]}$
     $f = x^3 + y^3 + z^3 + 3x^2y + 3xy^2 + 3y^2z + 3yz^2 + 3xz^2 + 3x^2z.$

INTRODUCTION

Properties of rationality

k a field, X/k projective integral variety

- X is **rational**: X is birational to $\mathbb{P}^n_k \Leftrightarrow k(X)/k$ is a purely transcendental extension;
- X is **stably rational**: $X \times \mathbb{P}^m_k$ is rational, for some m;
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We have implications $\Downarrow$.
All notions are equivalent for $X/\mathbb{C}$ smooth, of dimension 1 ($X \simeq \mathbb{P}^1_{\mathbb{C}}$) or 2 (birational class of $\mathbb{P}^2_{\mathbb{C}}$).
Next: typical examples and counterexamples.

Rationality proofs

Notation:

$X_d \subset \mathbb{P}^n_k: \ f(x_0, \ldots x_n) = 0, deg\ f = d$ a smooth hypersurface.

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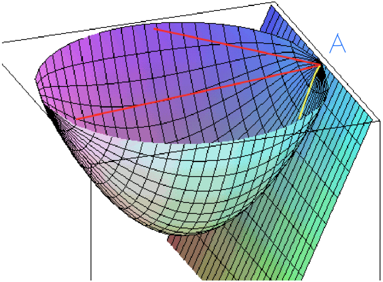
- smooth quadrics X_2 with $X_2(k) \neq \emptyset$ are **rational**:



Rational parametrization:
nontangent lines through $A \leftrightarrow$ second intersection point with the quadric.

Irrationality proofs over $\mathbb{C}$: classical

Classical methods:

- compute some invariant $i(X)$;
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Examples of not rational smooth threefolds:

1. $X_3 \subset \mathbb{P}^4_{\mathbb{C}}$ (Clemens-Griffiths, using *intermediate Jacobian*);
2. $X_4 \subset \mathbb{P}^4_{\mathbb{C}}$ (Iskovskikh-Manin, using *rigidity*);
3. Z a resolution of

$$Y : z_4^2 - f_4(x_0, x_1, x_2, x_3) = 0$$

   a double cover of $\mathbb{P}^3_{\mathbb{C}}$ ramified along some quartic (Artin-Mumford, $H^3(Z, \mathbb{Z})_{tors} = Br\, Z \neq 0$).

These varieties provide examples of unirational not rational complex threefolds.

Irrationality proofs over $\mathbb{C}$: specialization

(Beauville, Voisin, Colliot-Thélène–Pirutka, Totaro, Schreieder):

- consider a family of varieties:

$$\begin{array}{ccccc} \mathcal{X} & \longleftarrow & X_0 & \longleftarrow & \textit{reference variety} \\ \downarrow & & \downarrow & & \\ B & \longleftarrow & 0 & & \end{array}$$

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- (in some cases, all previously computable $i(X)$ vanish);
- $\mathcal{X}_b$ very general: $b \notin \cup_{i \in \mathbb{N}} B_i(\mathbb{C})$, $B_i \subset B$ closed.
- **EPSILON**:
  - restriction on singularities of X_0;
  - "restriction to subvarieties" for i (Schreieder).

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4. (Schreieder) $X_d \subset \mathbb{P}^{n+1}$ with

   $$d \geq log_2 n + 2,$$

   this generalizes previous bounds by Kollár, and Totaro, of order $d \sim\geq 2/3n$.

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Other examples:

- cyclic covers,
- complete intersections,
- hypersurfaces in $\mathbb{P}^m \times \mathbb{P}^n$, and more.

Available reference varieties X_0

- X_0 : a conic of quadric surface bundle over $\mathbb{P}^2$,

$$i = Br(X_0')[2] = H^2_{nr}(X_0, \mathbb{Z}/2) \subset H^2(\mathbb{C}(X_0), \mathbb{Z}/2)$$

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- X_0 : a fibration over $\mathbb{P}^n$ in Fermat-Pfister forms, $i = H^m_{nr}(X_0)$.

Galois cohomology

$H^i(K, \mathbb{Z}/2)$ and residues

Assume: $K \supset \mu_n$.

1. - $H^0(K, \mathbb{Z}/n) \simeq \mathbb{Z}/n$;
   - $H^1(K, \mathbb{Z}/n) \simeq K^*/K^{*n}$ (Kummer),
     for $a \in K^*$, we will still denote by a its class in $H^1(K, \mathbb{Z}/n)$.
   - $Br(K)[n] = H^2(K, \mathbb{Z}/n)$ (Kummer);
     symbols: $(a, b) := a \cup b \in H^2(K, \mathbb{Z}/n), a, b \in K^*$.

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2.
   - $v : K \to \mathbb{Z} \cup \infty$ a discrete valuation of rank 1:
     Recall: $v(x) = \infty \Leftrightarrow x = 0$
     $v(xy) = v(x) + v(y)$
     $v(x + y) \geq min(v(x), v(y))$
   - A be the valuation ring: $A = \{x, v(x) \geq 0\}$,
   - $\kappa(v)$ the residue field: $\kappa(v) = A/m$,
     $m = \{x, v(x) > 0\} = (\pi_A)$, π_A is a uniformizer

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   - this gives $\partial_v^i : H^i(K, \mathbb{Z}/n) \to H^{i-1}(\kappa(v), \mathbb{Z}/n)$.
   - ∂_v^i **factors through the completion** $H^i(K_v, \mathbb{Z}/n)$

Formulas for residus

$a, b \in H^1(K, \mathbb{Z}/n) \simeq K^*/K^{*n}$

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   where $\overline{\frac{a^{v(b)}}{b^{v(a)}}}$ is the image of the unit $\frac{a^{v(b)}}{b^{v(a)}}$ in $\kappa(v)^*/\kappa(v)^{*n}$.

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3. In particular, $\partial_v^2(a, b) = 0$ if $v(a) = v(b) = 0$.

Example

- $S = \mathbb{P}^2_{\mathbb{C}}$, $K = \mathbb{C}(x, y)$, $\alpha = (x, y) \in H^2(K, \mathbb{Z}/2)$;
- $v_D : K^* \to \mathbb{Z}$ is the order of vanishing at $D = \{x = 0\}$;
- recall: $\partial^2_v(a, b) = (-1)^{v(a)v(b)} \overline{\frac{a^{v(b)}}{b^{v(a)}}}$;
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H^i_{nr}: definition

- X/k an integral variety, then

$$H^2_{nr}(X) = H^2_{nr}(k(X)/k) = \cap_v \mathrm{Ker}\partial^2_v$$

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- Birational invariant by definition (Saltman, Bogomolov, Colliot-Thélène-Ojanguren).
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- Fact: if X is smooth and projective, $H^2_{nr}(X, \mathbb{Z}/n) \simeq Br(X)[n]$.

Strategy for fibrations (Colliot-Thélène - Ojanguren)

- Set up:

$$\begin{array}{ccc} X_K & \longrightarrow & X \longleftarrow \textit{fibration in geometrically rational varieties} \\ \downarrow & & \downarrow \pi \\ K & \longrightarrow & S = \mathbb{P}^2_{\mathbb{C}} \end{array}$$

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- $\alpha \in H^2(K)$ is ramified on S as $H^2_{nr}(\mathbb{C}(S)/\mathbb{C}) = H^2(\mathbb{C}) = 0$.
- idea: if $\partial^2_{v_D}(\alpha) \neq 0$, then π degenerates along D.

Relative unramified cohomology $H^i_{nr,\pi}(k(X)/k) \subset H^i(k(X))$

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Definition

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H^i_{nr,\pi}(k(X)/k) = \mathrm{Im}[H^i(K) \to H^i(K(X_K))] \bigcap
$$
$$
\cap_P \mathrm{Ker}[H^i(K) \to H^i(K_P) \to H^i(K_P(X_{K_P}))],
$$

where

- P runs over all scheme points of S of positive codimension: $P \in S^{(i)}$ for $i > 0$
- K_P is the field of fractions of the completed local ring $\widehat{O}_{S,P}$.

Properties

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- $H^i_{nr,\pi}(k(X)/k) \subset H^i_{nr}(k(X)/k)$.
- if $\alpha \in H^i_{nr,\pi}(k(X)/k)$ nonzero, then X is a *reference variety* (Schreieder).

Cubic surface bundles

Computing $H^2_{nr,\pi}$

$$H^2_{nr,\pi}(k(X)/k) = \mathrm{Im}[H^2(K) \to H^2(K(X_K))] \bigcap \cap_P \mathrm{Ker}[H^2(K) \to H^2(K_P) \to H^2(K_P(X_{K_P}))].$$

Let $Y = X_K$.

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$$H^2_{nr,\pi}(k(X)/k) = \mathrm{Im}[H^2(K) \to H^2(K(X_K))] \bigcap$$
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- not injective, and what is the kernel ($F = K_P$)?

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Known answers for:

- Y a quadric (Arason, Pfister, Kahn-Rost-Sujatha)
- Y a geometrically rational surface (Colliot-Thélène - Karpenko - Merkurjev).

Rational surfaces and kernels for $H^2(\cdot, \mathbb{Z}/3)$

(Colliot-Thélène - Karpenko - Merkurjev)
F a field, Y/F geometrically rational surface. Then

- $\mathrm{Ker}[H^2(F, \mathbb{Z}/3) \to H^2(F(Y), \mathbb{Z}/3)] \neq 0$ iff
- Y is F-birational to Y' a non-split Severi-Brauer (SB) surface.

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$$\mathrm{Ker}[H^2(F, \mathbb{Z}/3) \to H^2(F(Y), \mathbb{Z}/3)] \simeq \mathbb{Z}/3,$$

generated by the class of Y'.

Example: minimal cubic

$$Y : au^3 + bv^3 + abw^3 + ft^3 = 0, \; a, b, f \in F$$

- Assume: none of the elements a, b, ab, f, af, bf is a cube in F.
- (Segre) then the surface is minimal, and

$$H^2(F, \mathbb{Z}/3\mathbb{Z}) \to H^2(F(Y), \mathbb{Z}/3\mathbb{Z})$$

is injective.

Example: nonminimal cubic

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- if a is a cube in $F(Y)$, then $(a, b) = 0$.
- Otherwise, let $L = F(Y)(\sqrt[3]{a})$. In $F(Y)$ we have a relation

  $$b = -\frac{t^3 + au^3}{v^3 + aw^3},$$

  so

  $$b = N_{L/F(Y)}(\beta)$$

  where

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- Hence in $H^2(F(Y), \mathbb{Z}/3\mathbb{Z})$:

  $$(a, b) = (a, N_{L/F(Y)}(\beta)) = N_{L/F(Y)}(a, \beta) = 0.$$

Example

- $k = \mathbb{C}$
  (or k an algebraically closed field of $char(k) \neq 3$)
- $X \subset \mathbb{P}^2_{[x:y:z]} \times \mathbb{P}^3_{[u:v:w:t]}$ is a cubic surface bundle over k:

$$xz^2u^3 + y^2zv^3 + xy^2w^3 + ft^3 = 0,$$

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- Let $K = \mathbb{C}(\mathbb{P}^2) = \mathbb{C}(x/z, y/z)$, let
  $\alpha = (x/z, y/z) \in H^2(K, \mathbb{Z}/3)$. Then

$$\alpha \in H^2_{nr,\pi}(\mathbb{C}(X)/\mathbb{C}, \mathbb{Z}/3).$$

Sketch of proof: α nonzero in $C(X) = K(X_K)$

the generic fibre $Y = X_K$ of π is a minimal cubic surface:

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Recall:

$$au^3 + bv^3 + abw^3 + ft^3 = 0, \; a, b, f \in K$$

if none of the elements a, b, ab, f, af, bf is a cube then
$H^2(K, \mathbb{Z}/3) \to H^2(K(Y), \mathbb{Z}/3)$ is injective.

Sketch of proof: ramification of α

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Question: For which divisors $D \subset \mathbb{P}^2_{\mathbb{C}}$ one has $\partial_D(\alpha) \neq 0$?

Answer: $x = 0$ or $y = 0$ or $z = 0$.

Sketch of proof: α zero in $K_P(Y)$

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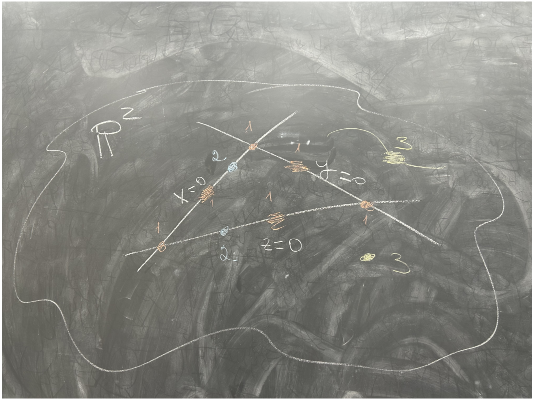
Let $P \in \mathbb{P}^2_k$ be a point of positive codimension. We have three cases:

1. P is the generic point of one of three lines $x = 0$, $y = 0$, or $z = 0$, or an intersection point of two of these lines.
2. P is a closed point lying on only one of the lines $x = 0$, $y = 0$, or $z = 0$.
3. All other cases.

Blackboard



Sketch of proof: α zero in $K_P(Y)$, case 1

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- Y_{K_P} is
$$\frac{x}{z}u^3 + \frac{y^2}{z^2}v^3 + \frac{x}{z}\frac{y^2}{z^2}w^3 + t^3 = 0$$
so that the element $(x/z, y^2/z^2) = 2\alpha$ is in the kernel of the map
$$H^2(K_P, \mathbb{Z}/3) \to H^2(K_P(Y), \mathbb{Z}/3\mathbb{Z}).$$

Sketch of proof: α zero in $K_P(Y)$, case 2

$$xz^2u^3 + y^2zv^3 + xy^2w^3 + ft^3 = 0, \ \alpha = (x/z, y/z).$$

P is a closed point lying on only one of the lines $x = 0$, $y = 0$, or $z = 0$.

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- enough: $\alpha = 0$ over K_P.
- assume: P is on the line $x = 0$:
- then y/z is a nonzero element in the residue field $\kappa(P) = \mathbb{C}$, hence a cube
- Hence y/z is a cube in K_P.

Sketch of proof: α zero in $K_P(Y)$, case 3

$$xz^2u^3 + y^2zv^3 + xy^2w^3 + ft^3 = 0, \ \alpha = (x/z, y/z).$$

P is not on the lines $x = 0$, $y = 0$, or $z = 0$.

- x/z and y/z are units in the local ring of P, so that the image of α in K_P comes from the cohomology group $H^2_{ét}(\widehat{\mathcal{O}}_{\mathbb{P}^2,P}, \mathbb{Z}/3)$.

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- x/z and y/z are units in the local ring of P, so that the image of α in K_P comes from the cohomology group $H^2_{\acute{e}t}(\widehat{\mathcal{O}}_{\mathbb{P}^2,P}, \mathbb{Z}/3)$.
- $H^2_{\acute{e}t}(\widehat{\mathcal{O}}_{\mathbb{P}^2,P}, \mathbb{Z}/3) = H^2(\kappa(P), \mathbb{Z}/3) = 0$ by cohomological dimension.

Corollary

We obtained:

$$xz^2u^3 + y^2zv^3 + xy^2w^3 + ft^3 = 0 \subset \mathbb{P}^2_{[x:y:z]} \times \mathbb{P}^3_{[u:v:w:t]}$$

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$$f = x^3 + y^3 + z^3 + 3x^2y + 3xy^2 + 3y^2z + 3yz^2 + 3xz^2 + 3x^2z$$

is a reference variety.

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is a reference variety.
Then:

Theorem (Krylov-Okada, Nicaise-Ottem)

Let k be an algebraically closed field of char $(k) \neq 3$. A very general hypersurface of bidegree $(3,3)$ in $\mathbb{P}^2_k \times \mathbb{P}^3_k$ is not stably rational.

General formula

$\pi : X \to S = \mathbb{P}^2_{\mathbb{C}}$ cubic surface bundle, $K = \mathbb{C}(x, y)$.

$$H^2_{nr,\pi}(\mathbb{C}(X)/\mathbb{C}, \mathbb{Z}/3) = \mathrm{Im}[H^2(K, \mathbb{Z}/3) \to H^2(K(X_K), \mathbb{Z}/3)] \bigcap$$
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- $\alpha \in H^2(K)$ is determined by residues at $P \in S$ of codimension 1, by Bloch-Ogus:

$$0 \to H^2(K, \mathbb{Z}/3) \stackrel{\oplus\partial^2}{\to} \oplus_{P \in S^{(1)}} H^1(\kappa(P), \mathbb{Z}/3) \to \stackrel{\oplus\partial^1}{\to} \oplus_{p \in S^{(2)}} H^0(\kappa(p), \mathbb{Z}/3)$$

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  X_{K_P} is birational to a SB surface $\Rightarrow$ the fiber
  $X_P = \cup 3$ conjugated planes (condition appeared in a joint work with A. Auel and C. Böhning).

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Set up: $\pi : X \to S = \mathbb{P}^2_{\mathbb{C}}$ cubic surface bundle, $K = \mathbb{C}(x, y)$.
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- $C = \cup_{i=1}^n C_i \subset S$ a divisor corresponding to the set of codimension 1 points of S over which the fibre of π is geometrically a union of three planes permuted by Galois.
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Assume C is snc. Then (briefly):

- $\alpha \in H^2_{nr,\pi}$ is only allowed to have residues γ_i at C_i + condition on K_P.
- glue by Bloch-Ogus.

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Then

$$H^2_{nr,\pi}(\mathbb{C}(X)/\mathbb{C}, \mathbb{Z}/3) = \{\underline{a} = \{a_i\}_{i=1}^n, a_i \in \{-1, 0, 1\}\} \subset (\mathbb{Z}/3)^n$$

(i) $a_i \neq 0 \Rightarrow X_{K_{C_i}}$ is birational to SB;

(ii) (Bloch-Ogus)

$$\sum_{i=1}^n \sum_{P \in S^{(2)}} \partial_P(\gamma_i^{a_i}) = 0$$

(iii) if $P \in C_i \cap C_j$ and if $\partial_P(\gamma_i^{a_i}) = -\partial_P(\gamma_j^{a_j}) \neq 0$, one has that the base change X_{K_P} is birational to SB.

The end

THANK YOU!!!